A generalization of a problem of Fremlin (Axiomatic Set Theory and Set-theoretic Topology)

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A generalization of a problem of Fremlin

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1 Fremlin-Miller Covering Principle

The following result is stated in A. Miller [3] as an answer to a question by David Fremlin:

Theorem 1. (Theorem 3.7 in A. Miller [3]) *The following holds in the generic extension obtained by adding at least $\aleph_3$ Cohen reals to a model of* CH*:*

(1.1) *For any family $\mathcal{F}$ of Borel sets with $|\mathcal{F}| = \aleph_2$ such that $\bigcap\mathcal{F} = \emptyset$, there is a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \leq \aleph_1$ such that $\bigcap\mathcal{F}' = \emptyset$.*

Note that by moving to complements of elements of $\mathcal{F}$, the assertion (1.1) can be also conceived as a covering property resembling Lindelöf property of topological spaces. Thus we shall call here the property (1.1) the Fremlin-Miller Covering Principle. More generally, for cardinals $\kappa \geq \lambda$, let us denote with FMCP(κ, λ) the following parametrized Fremlin-Miller Covering Principle:

FMCP(κ, λ): For any family $\mathcal{F}$ of Borel sets with $|\mathcal{F}| < \kappa$ such that $\bigcap\mathcal{F} = \emptyset$ there is $\mathcal{F}' \in [\mathcal{F}]^{<\lambda}$ such that $\bigcap\mathcal{F}' = \emptyset$.

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Lemma 2. ([3]) (0) *For cardinals* $\kappa \geq \kappa' \geq \lambda' \geq \lambda$, $\mathrm{FMCP}(\kappa, \lambda)$ *implies* $\mathrm{FMCP}(\kappa', \lambda')$.

(1) $\mathrm{FMCP}(\kappa, \kappa)$ *holds for any cardinal* κ.

(2) $\mathrm{FMCP}(\mathfrak{c}^+, \mathfrak{c})$ *does not hold.*

(3) $\mathrm{FMCP}(\aleph_2, \aleph_1)$ *does not hold.*

(4) *If* κ *is one of* $\mathfrak{a}$, $\mathfrak{b}$, ... *or* $\mathfrak{b}^*$ *then* $\mathrm{FMCP}(\kappa^+, \kappa)$ *does not hold.*

Proof. (0), (1): Trivial by definition.

(2): Let $\mathcal{A}$ be a maximal almost disjoint family $\subseteq [\omega]^{\aleph_0}$ of cardinality $\mathfrak{c}$. For each $a \in \mathcal{A}$, let

$$X_a = \{x \in \mathcal{P}(\omega) \,:\, x \text{ is almost disjoint from } a\}.$$

Then $X_a \in Borel(\mathcal{P}(\omega))$ for all $a \in \mathcal{A}$ and $\bigcap_{a\in\mathcal{A}} X_a = \emptyset$ by the maximality of $\mathcal{A}$ but $\bigcap_{a\in A'} X_a \neq \emptyset$ for any $A' \subsetneq A$.

(3): Let $\langle\langle f_\alpha\rangle_{\alpha<\omega_1}, \langle g_\beta\rangle_{\beta<\omega_1}\rangle$ be a Hausdorff gap. For each $\alpha < \omega_1$, let

$$X_\alpha = \{f \in {}^\omega\omega \,:\, f_\alpha \leq^* f \leq^* g_\alpha\}.$$

Then X_α's are Borel sets and $\bigcap_{\alpha<\omega_1} X_\alpha = \emptyset$ but $\bigcap_{\alpha\in I} X_\alpha \neq \emptyset$ for any countable $I \subseteq \omega_1$.

(4): Similarly to (2) and (3). □ (Lemma 2)

By Lemma 2, "$\aleph_2 < \kappa \leq \mathfrak{c}$ and $\mathrm{FMCP}(\kappa, \aleph_2)$" is the first non-trivial instance of the principle $\mathrm{FMCP}(\kappa, \lambda)$.

It is easy to show that the following principle for cardinals $\kappa \leq \lambda$ is a generalization of the corresponding parametrized Fremlin-Miller Covering Principle:

> $\mathrm{GFMCP}(\kappa, \lambda)$: For any projective relation $R \subseteq \mathbb{R}^2$, and $X \in [\mathbb{R}]^{<\kappa}$, if X is unbounded in $\langle\mathbb{R}, R\rangle$, there is $X_0 \in [X]^{<\lambda}$ such that X_0 is unbounded in $\langle\mathbb{R}, R\rangle$.

Here we say X is unbounded in $\langle\mathbb{R}, R\rangle$ if

$$\forall r \in \mathbb{R}\; \exists x \in X\; \neg(x \, R \, r)$$

holds.

Proposition 3. $\mathrm{GFMCP}(\kappa, \lambda)$ *implies* $\mathrm{FMCP}(\kappa, \lambda)$ *for any cardinals* $\kappa \geq \lambda$.

Proof. Assume that $\mathrm{GFMCP}(\kappa, \lambda)$ holds and suppose that $\langle X_\alpha \,:\, \alpha < \delta\rangle$ is a sequence of Borel subsets of $\mathbb{R}$ for some $\delta < \kappa$ such that $\bigcap_{\alpha<\delta} X_\alpha = \emptyset$.

For $\alpha < \delta$, let c_α be a Borel code of X_α and let $X^* = \{c_\alpha \,:\, \alpha < \delta\}$.

For any $x \in \mathbb{R}$, let

$$(1.2) \quad B_x = \begin{cases} \text{the Borel set coded by } x, & \text{if } x \text{ is a Borel code} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $R \subseteq \mathbb{R}^2$ be defined by

$$x \mathrel{R} y \iff B_y \text{ is a non empty subset of } B_x$$

for x, $y \in \mathbb{R}$. The relation R is easily seen to be Π^1_1. Clearly, we have

$$(1.3) \quad X \text{ is unbounded in } \langle \mathbb{R}, R\rangle \iff \bigcap\{B_x : x \in X\} = \emptyset$$

for any $X \subseteq \mathbb{R}$. In particular, X^* above is unbounded in $\langle \mathbb{R}, R\rangle$. By GFMCP$(\kappa, \lambda)$, there is $X^{**} \subseteq X^*$ of cardinality $< \lambda$ such that X^{**} is already unbounded in $\langle \mathbb{R}, R\rangle$. Thus, again by (1.3), $\bigcap_{\alpha\in I} X_\alpha = \emptyset$ for $I = \{\alpha < \delta : c_\alpha \in X^{**}\}$. □ (Proposition 3)

The proof of Theorem 1 in [3] can be recast to show the following consistency result on GFMCP$(\mathfrak{c}, \aleph_2)$:

Theorem 4. *Let $\kappa < \mu$ be regular cardinals. Suppose that $\mathbb{P}_{\{\alpha\}}$, $\alpha < \mu$ are posets such that*

(1.4) $\mathbb{P}_{\{\alpha\}} \cong \mathbb{P}_{\{0\}}$ *for all* $\alpha < \mu$;

(1.5) $\mathbb{P} = \prod^{fin}_{\alpha<\mu} \mathbb{P}_\alpha$ *satisfies the c.c.c.;*

(1.6) $|\,\mathbb{P}_{\{0\}}\,| \le \kappa = \kappa^{\aleph_0}, \kappa^+ < \mu$.

Then $\Vdash_{\mathbb{P}}$ "GFMCP(μ, κ^+)".

We shall give the details of the proof of Theorem 4 in the next section.

The formulation of GFMCP$(\kappa, \aleph_2)$ has a certain resemblance to that of HP$(\aleph_2)$ of J. Brendle and S. Fuchino [1]. This feeling is also supported by the fact that they both hold in Cohen models. The following proposition shows however that these principles are rather independent to each other:

Proposition 5. (1) $\mathfrak{c} \ge \aleph_3 \wedge$ GFMCP$(\mathfrak{c}, \aleph_2) \wedge \neg$HP$(\aleph_2)$ *is consistent.*

(2) $\neg$GFMCP$(\aleph_3, \aleph_2) \wedge$ HP$(\aleph_2)$ *is consistent.*

Proof. (1): The arguments used in the proof of Theorem 4 are also valid for the generic extension with (measure theoretic) side-by-side product of random forcing. It is known that HP$(\aleph_2)$ does not hold in a random extension (see [1]).

(2): In a model of HP$(\aleph_2) \wedge \mathfrak{c} = \aleph_2$ we have $\neg$GFMCP$(\aleph_3, \aleph_2)$ by Lemma 2, (2). □ (Proposition 5)

Problem 1. *Is* $\neg$GFMCP$(\mathfrak{c}, \aleph_2) \wedge$ HP$(\aleph_2)$ *consistent under* $\mathfrak{c} \ge \aleph_3$ *?*

2 Proof of the consistency result

In this section we prove Theorem 4.

Let $\kappa < \mu$ be regular cardinals and $\mathbb{P}_{\{\alpha\}}$, $\alpha < \mu$ satisfy (1.4), (1.5) and (1.6). For $X \subseteq \mu$, we denote

(2.1) $\quad \mathbb{P}_X = \prod^{fin}_{\alpha \in X} \mathbb{P}_\alpha.$

Thus $\mathbb{P} = \mathbb{P}_\mu$. We assume that finite support product is introduced just as in [1]. In particular, we have $\mathbb{P}_X \lessdot \mathbb{P}_Y \lessdot \mathbb{P}$ for all $X \subseteq Y \subseteq \mu$.

A bijection $f : \mu \to \mu$ induces an automorphism of $\mathbb{P}$ and this induces in turn an automorphism on $\mathbb{P}$-names. We shall denote both of these automorphisms by $\tilde{f}$.

All of the following Lemmas 6, 7 and 8 are folklore:

Lemma 6. *Suppose that $X \subseteq \mu$ and $\dot{x}_\xi$, $\xi < \delta$ are $\mathbb{P}$-names of elements of $\mathcal{H}(\aleph_1)$ (in the sense of $V^{\mathbb{P}}$) such that* $\mathrm{supp}(\dot{x}_\xi) \subseteq X$ *for all $\xi < \delta$. If*

(2.2) $\quad X \setminus \bigcup\{\mathrm{supp}(\dot{x}_\xi) : \xi < \delta\}$ *is uncountable,*

then we have

(2.3) $\quad \Vdash_{\mathbb{P}}$ "$\langle \mathcal{H}(\aleph_1)^{V[\dot{G} \cap \mathbb{P}_X]}, \{\dot{x}_\xi : \xi < \delta\}, \ldots, \in\rangle \prec \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \in\rangle$".

Proof. Suppose that $p \Vdash_{\mathbb{P}}$ "$\langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \in\rangle \models \exists x \varphi(x, \dot{a}_1, \ldots, \dot{a}_n)$" for a $\mathcal{L}_{\mathrm{ZF}}$-formula φ and $\mathbb{P}_X$-names $\dot{a}_1, \ldots, \dot{a}_n$ of elements of $\mathcal{H}(\aleph_1)$. By the Tarski-Vaught criterion, it is enough to show that

$$p \Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \in\rangle \models \varphi(\dot{c}, \dot{a}_1, \ldots, \dot{a}_n) \text{”}$$

for some $\mathbb{P}_X$-name $\dot{c}$ of an element of $\mathcal{H}(\aleph_1)$.

By (1.5), we may assume without loss of generality that

(2.4) $\quad \mathrm{supp}(\dot{a}_1), \ldots, \mathrm{supp}(\dot{a}_n)$ are all countable.

By (2.2), we may assume that $\mathrm{supp}(p) \subseteq X$. Let

(2.5) $\quad X' = \bigcup\{\mathrm{supp}(\dot{x}_\xi) : \xi < \delta\} \cup \bigcup\{\mathrm{supp}(\dot{a}_i) : i \in n+1 \setminus 1\} \cup \mathrm{supp}(p).$

By the assumptions above, we have $X' \subseteq X$. By (2.2) and (2.4), $X \setminus X'$ is still uncountable. By Maximal Principle, there is a $\mathbb{P}$-name $\dot{b}$ of an element of $\mathcal{H}(\aleph_1)$ such that

$$p \Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \in\rangle \models \varphi(\dot{b}, \dot{a}_1, \ldots, \dot{a}_n) \text{”}.$$

By (1.5), we can find such $\dot{b}$ with countable $\mathrm{supp}(\dot{b})$.

Let $f : \mu \rightarrow \mu$ be a bijection such that

$$f \restriction X' = id_{X'} \text{ and } f''\mathrm{supp}(\dot{b}) \subseteq X.$$

Let $\dot{c} = \tilde{f}(\dot{b})$. Then $\dot{c}$ is a $\mathbb{P}$-name and

$$p \Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \varphi(\dot{c}, \dot{a}_1, ..., \dot{a}_n) \text{”}.$$

□ (Lemma 6)

Lemma 7. *Suppose that $X \subseteq \mu$, $\mu \setminus X$ is infinite and $X_0 \subseteq \mu \setminus X$ is countable. Let $\dot{x}_\xi$, $\xi < \delta$ be $\mathbb{P}$-names of elements of $\mathcal{H}(\aleph_1)$ (in the sense of $V^{\mathbb{P}}$) such that* $\mathrm{supp}(\dot{x}_\xi) \subseteq X$ *for all $\xi < \delta$.*

If $p \Vdash_{\mathbb{P}}$ “$\langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \varphi$” for some $p \in \mathbb{P}_X$ and $\mathcal{L}_{\mathrm{ZF}}$-sentence φ then we have $p \Vdash_{\mathbb{P}_{X \cup X_0}}$ “$\langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \varphi$”.

Thus we have

$$\Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1)^{V[G \cap (X \cup X_0)]}, \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \equiv \langle \mathcal{H}(\aleph_1)^{V[G]}, \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \text{”}.$$

Proof. It is enough to show the following $(2.6)_\psi$ for all $\mathcal{L}_{\mathrm{ZF}}$-formula $\psi = \psi(x_1, ..., x_n)$ by induction on ψ:

$(2.6)_\psi$ For any $\mathbb{P}$-names $\dot{a}_1, ..., \dot{a}_n$ of elements of $\mathcal{H}(\aleph_1)$ such that

(2.6a) $\mathrm{supp}(\dot{a}_i) \subseteq X \cup X_0$ for $i \in n+1 \setminus 1$ and

(2.6b) $X_0 \setminus \bigcup \{\mathrm{supp}\, \dot{a}_i : i \in n+1 \setminus 1\}$ is infinite,

if $q \in \mathbb{P}_{X \cup X_0}$ and $q \leq_{\mathbb{P}} p$, then

$$q \Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \psi(\dot{a}_1, ..., \dot{a}_n) \text{”}$$

if and only if

$$q \Vdash_{\mathbb{P}_{X \cup X_0}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \psi(\dot{a}_1, ..., \dot{a}_n) \text{”}.$$

The crucial step in the induction proof of $(2.6)_\psi$ is when $\psi(x_1, ..., x_n)$ is of the form $\exists x \eta(x, x_1, ..., x_n)$.

Suppose that $\dot{a}_1, ..., \dot{a}_n$ are $\mathbb{P}$-names of elements of $\mathcal{H}(\aleph_1)$ satisfying (2.6a) and (2.6b), $q \in \mathbb{P}_{X \cup X_0}$, $q \leq_{\mathbb{P}} p$ and

$$q \Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \psi(\dot{a}_1, ..., \dot{a}_n) \text{”}.$$

Then there is a $\mathbb{P}$-name $\dot{a}$ of an element of $\mathcal{H}(\aleph_1)$ such that

$$q \Vdash_{\mathbb{P}} \text{“} \langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in \rangle \models \eta(\dot{a}, \dot{a}_1, ..., \dot{a}_n) \text{”}.$$

By (1.5), we may assume that $\mathrm{supp}(\dot{a})$ is countable. Let $f : \mu \to \mu$ be a bijection such that

(2.7) $\quad f \restriction X' = id_{X'}$

where $X' = X \cup \bigcup\{\mathrm{supp}(\dot{a}_i) : i \in n+1 \setminus 1\} \cup \mathrm{supp}(q)$;

(2.8) $\quad f''(\mathrm{supp}(r) \cup \mathrm{supp}(\dot{a})) \subseteq X \cup X_0$ and

(2.9) $\quad X_0 \setminus (\bigcup\{\mathrm{supp}(\dot{a}_i) : i \in n+1 \setminus 1\} \cup \mathrm{supp}(\dot{a}))$ is infinite.

Then by induction's hypothesis, we have

$$q \Vdash_{\mathbb{P}_{X \cup X_0}} \text{“}\langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in\rangle \models \eta(\tilde{f}(\dot{a}), \dot{a}_1, ..., \dot{a}_n)\text{”}.$$

It follows that

$$q \Vdash_{\mathbb{P}_{X \cup X_0}} \text{“}\langle \mathcal{H}(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, ..., \in\rangle \models \psi(\dot{a}_1, ..., \dot{a}_n)\text{”}.$$

The “only if” direction of this induction step can be shown similarly and more easily. $\square$ (Lemma 7)

If G is a $(V, \mathbb{Q})$-generic set for a poset $\mathbb{Q}$ and M is a set, we denote with $M[G]$ the set $\{\dot{x}^G : \dot{x} \in V^{\mathbb{Q}} \cap M\}$.

Lemma 8. *Suppose that $\mathbb{Q}$ is a poset and $\mathbb{P} \in M \prec \mathcal{H}(\theta)$ for sufficiently large regular θ. If G is a $(V, \mathbb{Q})$-generic set then we have*

(2.10) $\quad M[G] \prec \mathcal{H}(\theta)[G]$.

Proof. Note that $\mathcal{H}(\theta)[G] = \mathcal{H}(\theta)^{V[G]}$. We check again the forcing version of Tarski-Vaught criterion.

Suppose that

(2.11) $\quad p \Vdash_{\mathbb{Q}} \text{“}\mathcal{H}(\theta) \models \exists x \varphi(x, \dot{a}_1, ..., \dot{a}_n)\text{”}$

for $\mathcal{L}_{\mathrm{ZF}}$-formula φ and $\mathbb{Q}$-names $\dot{a}_1, ..., \dot{a}_n$ of elements of M. We may assume that $\dot{a}_1, ..., \dot{a}_n \in M$. (2.11) is equivalent to

$$\mathcal{H}(\theta) \models p \Vdash_{\mathbb{Q}} \text{“}\exists x \varphi(x, \dot{a}_1, ..., \dot{a}_n)\text{”}.$$

Then by elementarity we have

$$M \models p \Vdash_{\mathbb{Q}} \text{“}\exists x \varphi(x, \dot{a}_1, ..., \dot{a}_n)\text{”}.$$

It follows that there is some $\dot{a} \in V^{\mathbb{P}} \cap M$ such that $M \models p \Vdash_{\mathbb{Q}}$ "$\varphi(\dot{a}, \dot{a}_1, ..., \dot{a}_n)$". By elementarity of M this is equivalent to $\mathcal{H}(\theta) \models p \Vdash_{\mathbb{Q}}$ "$\varphi(\dot{a}, \dot{a}_1, ..., \dot{a}_n)$" This, in turn, is equivalent to $p \Vdash_{\mathbb{Q}}$ "$\mathcal{H}(\theta) \models \varphi(\dot{a}, \dot{a}_1, ..., \dot{a}_n)$". □ (Lemma 8)

Proof of Theorem 4: Suppose that κ, μ, $\mathbb{P}_{\{\alpha\}}$, $\alpha < \mu$, $\mathbb{P}$ are as in Theorem 4, $p \in \mathbb{P}$ and

$$(2.12)\quad \begin{aligned} p \Vdash_{\mathbb{P}} \text{“}\{\dot{x}_\alpha : \alpha < \delta\} &\text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to} \\ R &= \{\langle x, y\rangle : \mathcal{H}(\aleph_1) \models \varphi(x, y, \dot{a})\}\text{”} \end{aligned}$$

where $\delta \leq \kappa$, φ is a $\mathcal{L}_{\mathrm{ZF}}$-formula and $\dot{a}$ is a $\mathbb{P}$-name of an element of $\mathcal{H}(\aleph_1)$.

Let $X \subseteq \lambda$ be such that $X \supseteq \bigcup\{\mathrm{supp}(\dot{x}_\alpha) : \alpha < \delta\} \cup \mathrm{supp}(p) \cup \mathrm{supp}(\dot{a})$. Then $|X| < \kappa$ and $X \setminus \{\mathrm{supp}(\dot{x}_\alpha) : \alpha < \delta\}$ is uncountable.

Let G be a $(V, \mathbb{P}_X)$-generic filter with $p \in G$ and let θ be a sufficiently large regular cardinal. By Lemma 7, we have

$$(2.13)\quad \mathcal{H}(\theta)[G] \models \Vdash_{\mathbb{P}_\omega} \text{“}\{\dot{x}^G_\alpha : \alpha < \delta\} \text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to } R\text{”}.$$

Let $M \prec \mathcal{H}(\theta)$ be such that

(2.14) $\mathbb{P}$, $\{\dot{x}_\alpha : \alpha < \delta\} \in M$;

(2.15) $[M]^{\aleph_0} \subseteq M$; and

(2.16) $|M| \leq \kappa$.

The last two conditions are possible since $\kappa^{\aleph_0} = \kappa$. By Lemma 8, we have

$$(2.17)\quad M[G] \prec \mathcal{H}(\theta)[G]$$

and hence

$$(2.18)\quad M[G] \models \Vdash_{\mathbb{P}_\omega} \text{“}\{\dot{x}^G_\alpha : \alpha < \delta\} \text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to } R\text{”}.$$

Note that $\mathbb{P}_\omega$ is an element of M but not $\mathbb{P}_{\mu \setminus Y}$ for Y as below and thus we cannot apply the elementary submodel argument to the latter poset.

Let $Y = \delta \cap M$. Since $|Y| \leq \kappa$ by (2.16), it is enough to show the following claim:

Claim 8.1. $\mathcal{H}(\theta)[G] \models \Vdash_{\mathbb{P}_{\mu \setminus X}}$ "$\{\dot{x}^G_\alpha : \alpha \in Y\}$ *is unbounded in* $\mathcal{H}(\aleph_1)$ *with respect to* R".

$\vdash$ In the following we work always in $\mathcal{H}(\theta)[G]$. Suppose that $q \in \mathbb{P}_{\mu \setminus X}$ and $\dot{x}$ is a $\mathbb{P}_{\mu \setminus X}$-name of an element of $\mathcal{H}(\aleph_1)$. Let $Z = \mathrm{supp}(\dot{x}) \cup \mathrm{supp}(p)$. Let $X_0 \in M$ be a countable subset of μ disjoint from $Y \cup Z$. $f : \mu \setminus X \to \mu \setminus X$ be a bijection such that

(2.19) $f''Z \subseteq Y \cup X_0$ and $f \restriction Y = id_Y$.

Note that $\tilde{f}(\dot{x})$ is a $\mathbb{P}_{X_0}$-name of an element of $\mathcal{H}(\aleph_1)$. By (1.5) and (2.15), we may assume that $\tilde{f}(\dot{x}) \in M$. Also note that $\mathbb{P}_{X_0} \cong \mathbb{P}_\omega$.

By (2.18), there are $\tilde{r} \leq_{\mathbb{P}_{X_0}} \tilde{f}(q)$ and $\alpha^* \in \delta \cap M (= Y)$ such that

(2.20) $M[G] \models \tilde{r} \Vdash_{\mathbb{P}_{X_0}}$ "$\neg(\dot{x}^G_{\alpha^*}\ R\ \tilde{f}(\dot{x}))$".

By (2.17), it follows that $\tilde{r} \Vdash_{\mathbb{P}_{X_0}}$ "$\neg(\dot{x}^G_{\alpha^*}\ R\ \tilde{f}(\dot{x}))$".

By Lemma 6, it follows that

(2.21) $\tilde{r} \Vdash_{\mathbb{P}_{\mu \setminus X}}$ "$\neg(\dot{x}^G_{\alpha^*}\ R\ \tilde{f}(\dot{x}))$".

Let $r = \tilde{f}^{-1}(\tilde{r})$. Then $r \leq_{\mathbb{P}_{\mu \setminus X}} q$. By mapping the parameters in (2.21) by $\tilde{f}^{-1}$, we obtain

(2.22) $r \Vdash_{\mathbb{P}_{\mu \setminus X}}$ "$\neg(\dot{x}^G_{\alpha^*}\ R\ \dot{x})$".

Since q and $\dot{x}$ were arbitrary, it follows that

(2.23) $\Vdash_{\mathbb{P}_{\mu \setminus X}}$ "$\{\dot{x}^G_\alpha : \alpha \in Y\}$ is unbounded in $\mathcal{H}(\aleph_1)$ with respect to R".

$\dashv$ (Claim 8.1)

$\square$ (Theorem 4)

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